

# A Non-Singular Black Hole

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## Abstract

We present a completely integrable deformation of the CGHS dilaton gravity model in two dimensions. The solution is a singularity free black hole that at large distances asymptotically joins to the CGHS solution.

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# 1 Introduction

One of the fundamental unsolved problems in theoretical physics is the unification of quantum theory and gravity. Many reasons why this has proved so difficult stem from the complicated nonlinear structure of the equations of general relativity. Gravitational equations are much simpler in lower dimensions. This is the reason why there has been so much activity related to the quantization of gravity in two and three dimensions. [1]-[13]. One of the most important results in 2d was the exactly solvable dilaton gravity model constructed by Callan, Giddings, Harvey and Strominger. The CGHS model has 2d black hole solutions that are remarkably similar to the Schwarzschild solution of general relativity.

Of the four fundamental interactions in nature, gravity is by far the weakest [14]-[15]. For this reason, we can hope to see quantum effects only in the vicinity of classical singularities. Penrose and Hawking [16]-[18] have shown that these singularities are endemic in general relativity. The general belief is that quantization will rid gravitation of singularities, just as in atomic physics it got rid of the singularity of the Coulomb potential. If this is indeed the case, then there must exist a non-singular gravitational effective action whose classical equations encode the full quantum theory. This effective action must have the Planck length  $L_{\text{Planck}}$  in it as an input parameter. For  $L \gg L_{\text{Planck}}$  the effective model must be indistinguishable from the classical gravity action. In an interesting series of papers [19]-[22] Brandenberger, Mukhanov and their collaborators have initiated a search for such effective models of 2d dilaton gravity. They investigated a procedure by which one could make models free of singularities. This parallels Landau's treatment of phase transitions in ferromagnets. Landau chose (the simplest) effective action (Gibbs potential in statistical mechanics parlance) that led to a qualitatively correct description of phase transitions.

A recent success in the field of 2d dilaton gravity has been the work of Louis-Martinez and Kunstatler [23], who reduced the solution of the general dilaton gravity model to the solution of two ordinary integrals, i.e. to two quadratures. In this paper we will use their result to construct an exactly solvable class of models — deformed CGHS models. For  $L \gg L_{\text{Planck}}$  these models go over into the CGHS model. Like CGHS, the deformed models are exactly solvable (the two quadrature integrals can be calculated in closed form) and have black hole solutions (solutions with event horizons). Unlike CGHS, the deformed models are non-singular. As we shall see, the maximal curvature is proportional to  $\frac{1}{L_{\text{Planck}}}$ .

## 2 CGHS Model

The action of all dilaton gravity models can be put into the general form

$$S = \int d^2x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi) + D(\phi) R \right] . \quad (1)$$

The potentials  $V(\phi)$  and  $D(\phi)$  classify all the possible models. Let us perform a conformal scaling of the metric

$$\tilde{g}_{\alpha\beta} = e^{-2F(\phi)} g_{\alpha\beta} , \quad (2)$$

where the scaling factor  $F(\phi)$  satisfies

$$\frac{dF}{d\phi} = -\frac{1}{4} \left( \frac{dD}{d\phi} \right)^{-1} . \quad (3)$$

This puts the action into the simplified form

$$S = \int d^2x \sqrt{-\tilde{g}} \left[ \tilde{\phi} \tilde{R} - \tilde{V}(\tilde{\phi}) \right] , \quad (4)$$

where  $\tilde{R}$  is the scalar curvature corresponding to  $\tilde{g}_{\alpha\beta}$ , and we have introduced the new dilaton field and potential according to

$$\tilde{\phi} = D(\phi) \quad (5)$$

$$\tilde{V}(\tilde{\phi}) = e^{2F(\phi)} V(\phi) . \quad (6)$$

This form of the dilaton gravity action is obviously much easier to work with since we have lost the kinetic term for the dilaton field.

A well known property of two dimensional manifolds allows us to locally, i.e. patch by patch, choose conformally flat coordinates for which

$$\tilde{g}_{\alpha\beta} = e^{2\rho} \eta_{\alpha\beta} . \quad (7)$$

Louis-Martinez and Kunstatter [23] have shown that we can choose a coordinate system in which the solution of the general dilaton model is static and given by

$$x = -2 \int \frac{d\tilde{\phi}}{W(\tilde{\phi}) + C} \quad (8)$$

$$e^{2\rho} = -\frac{C + W(\tilde{\phi})}{4} , \quad (9)$$

where the pre-potential  $W(\tilde{\phi})$  is given by  $\frac{dW}{d\tilde{\phi}} = \tilde{V}(\tilde{\phi})$ , and  $C$  is an invariant. As we can see, the solution is given in terms of two quadratures: equations (3) and (8), determining  $F(\phi)$  and  $\tilde{\phi}(x)$  respectively. A given model is completely integrable only if we can calculate both quadratures in closed form.

The CGHS model is an example of a completely integrable dilaton gravity model. The standard form of the CGHS action is

$$S = \int d^2x \sqrt{-g} e^{-2\varphi} (R + 4g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + 4\lambda^2) . \quad (10)$$

The simple field redefinition  $\phi = \sqrt{8} e^{-\varphi}$  puts this into the general form for dilaton gravity actions given in (1). We find

$$S = \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{2} \lambda^2 \phi^2 + \frac{1}{8} R \phi^2 \right) , \quad (11)$$

hence

$$V(\phi) = -\frac{1}{2} \lambda^2 \phi^2 \quad (12)$$

$$D(\phi) = \frac{1}{8} \phi^2 . \quad (13)$$

The first quadrature is easily integrated and we get

$$F(\phi) = -\ln \phi . \quad (14)$$

Using this,  $\tilde{\phi} = D(\phi)$ , as well as the definition of  $\tilde{V}(\tilde{\phi})$  we immediately find that  $\tilde{V}(\tilde{\phi}) = -\frac{1}{2} \lambda^2$ , and thus the pre-potential is  $\tilde{W}(\tilde{\phi}) = -\frac{1}{2} \lambda^2 \tilde{\phi}$ . The simplified form of the CGHS action is therefore

$$S = \int d^2x \sqrt{-\tilde{g}} \left( \tilde{\phi} \tilde{R} + \frac{1}{2} \lambda^2 \right) . \quad (15)$$

The CGHS model is completely integrable. In our notation this means that the second quadrature (8) can also be solved in closed form. A trivial integration gives

$$x = \frac{4}{\lambda^2} \ln \left( \frac{1}{2} \lambda^2 \tilde{\phi} - C \right) . \quad (16)$$

Inverting this we find

$$\tilde{\phi}(x) = \frac{2}{\lambda^2} \left( e^{\frac{\lambda^2}{4} x} + C \right) . \quad (17)$$

According to the general prescription this gives

$$\rho(x) = -\ln 2 + \frac{\lambda^2}{8}x \quad (18)$$

$$\phi(x) = \frac{4}{\lambda} \left( e^{\frac{\lambda^2}{4}x} + C \right)^{\frac{1}{2}} . \quad (19)$$

This, along with the expression for  $F(\phi)$ , gives us

$$F(x) = -\ln \frac{4}{\lambda} - \frac{1}{2} \ln \left( e^{\frac{\lambda^2}{4}x} + C \right) . \quad (20)$$

The scalar curvature of the general model can be given in terms of  $\rho(x)$  and  $F(x)$ . We find

$$R = -2 e^{-2(F+\rho)} \frac{d^2}{dx^2} (F + \rho) . \quad (21)$$

For the CGHS model this gives

$$R = \frac{4\lambda^2 C}{e^{\frac{\lambda^2}{4}x} + C} . \quad (22)$$

Obviously  $R$  has a singularity for  $C < 0$ . This is the CGHS black hole solution. In fact, it can be shown that  $-C$  is proportional to the mass, and hence  $C$  must be negative. From now on we will choose  $C = -1$ , thus putting the singularity at  $x = 0$ .

For later convenience we write the curvature as

$$R = -\frac{32}{A} , \quad (23)$$

where we have introduced

$$A = \frac{8}{\lambda^2} \left( e^{\frac{\lambda^2}{4}x} - 1 \right) . \quad (24)$$

The metric for the general dilaton model, given in terms of  $F$  and  $\rho$ , is simply

$$ds^2 = e^{2(F+\rho)} (-dt^2 + dx^2) . \quad (25)$$

In the case of CGHS we get

$$e^{2(F+\rho)} = \frac{\lambda^2}{64} \frac{e^{\frac{\lambda^2}{4}x}}{e^{\frac{\lambda^2}{4}x} - 1} , \quad (26)$$

which vanishes for  $x = -\infty$ . For stationary metrics the equation  $g_{00} = 0$  determines the horizon. Therefore, in these coordinates the CGHS black hole has a horizon at  $x = -\infty$ . The curvature, on the other hand, is well behaved at this point. As with the Schwarzschild black hole one can now find coordinates which are well behaved at the horizon. In this way one finally obtains information about the global character of the manifold.

### 3 Deformed CGHS Model

In this section we will construct a new dilaton gravity model that satisfies the following requirements:

1. It is completely integrable, i.e. both quadratures can be solved in closed form.
2. For  $x \rightarrow \infty$  it goes over into the CGHS model.
3. It is singularity free.

As we have seen, dilaton gravity models are specified by giving the two potentials  $D(\phi)$  and  $V(\phi)$ . It is very difficult to see how one should deform these potentials from their CGHS form in order to satisfy the above criteria. Note, however, that the models are also uniquely determined by giving  $F(\phi)$  and  $\tilde{V}(\tilde{\phi})$ . This is much better for us since we have now untangled the two integrability requirements:  $F(\phi)$  determines the first quadrature and  $\tilde{V}(\tilde{\phi})$  the second. Deformations of a given model correspond to changes of both of these functions. In this paper we will look at a simpler problem. We shall keep  $\tilde{V}(\tilde{\phi})$  fixed, i.e. it will have the same value as in the CGHS model

$$\tilde{V}(\tilde{\phi}) = -\frac{1}{2} \lambda^2 . \quad (27)$$

We will only deform  $F(\phi)$ . By doing this we are guaranteed that the second (and more difficult) quadrature is automatically solved. Because of this  $\tilde{\phi}(x)$  and  $\rho(x)$  are the same as in the CGHS model. Using the value for  $\rho(x)$  we may write the scalar curvature for all the remaining models solely in terms of  $F(x)$ . We have

$$R = -8 e^{-\frac{\lambda^2}{4} x} \left( e^{-2F} \frac{d^2 F}{dx^2} \right) . \quad (28)$$

Let us now choose  $F$ . From our second requirement we see that for large  $x$  the dilaton field  $\phi(x)$  must be near to its CGHS form. Specifically,  $x \rightarrow \infty$  corresponds to  $\phi \rightarrow \infty$ . Thus, our second requirement imposes that for  $\phi \rightarrow \infty$  we have

$$F(\phi) \rightarrow -\ln \phi . \quad (29)$$

$F(\phi)$  must also be such that the first quadrature (3) is exactly solvable. To do this we choose

$$F(\phi) = -\frac{1}{\alpha} \ln \left( \frac{1 + \beta \phi^\alpha}{\beta} \right) , \quad (30)$$

with  $\alpha > 0$ . The  $\alpha$  and  $\beta$  values parametrize our class of deformations. The first quadrature now gives

$$D(\phi) = \begin{cases} \frac{1}{8}\phi^2 + \frac{1}{4\beta} \ln \phi & \text{for } \alpha = 2 \\ \frac{1}{8}\phi^2 + \frac{1}{4\beta(2-\alpha)}\phi^{2-\alpha} & \text{for } \alpha \neq 2. \end{cases} \quad (31)$$

On the other hand, the potential  $V(\phi)$  is now simply

$$V(\phi) = -\frac{1}{2}\lambda^2 \left( \frac{1 + \beta\phi^\alpha}{\beta} \right)^{\frac{2}{\alpha}}. \quad (32)$$

The choice of  $\alpha$  corresponds to a choice of explicit model, while  $\beta$  just sets a scale for the dilaton field. Rather than work here with the general deformed model we will now concentrate on the simplest model in this class; the one corresponding to the choice  $\alpha = 4$ . The action for this model is

$$S = \int d^2x \sqrt{-g} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{2} \lambda^2 \left( \frac{1 + \beta\phi^4}{\beta} \right)^{\frac{1}{2}} + \frac{1}{8} \left( \phi^2 - \frac{1}{\beta\phi^2} \right) R \right). \quad (33)$$

Note that for  $\beta \rightarrow \infty$  this goes over into the action of the CGHS model. As we have seen,  $\beta$  is just a scale for  $\phi$ , hence, this is just a re-statement of our second requirement. From our construction we see that (33) corresponds, for each finite value of  $\beta$ , to a model that satisfies our first two requirements. All that is left is to check that the theory is indeed free of singularities. Being in two dimensions all that we need to check is the scalar curvature.

From (5) and (31) for  $\alpha = 4$  we find the connection between  $\phi$  and  $\tilde{\phi}$

$$\tilde{\phi} = \frac{1}{8} \left( \phi^2 - \frac{1}{\beta\phi^2} \right). \quad (34)$$

On the other hand, as we have seen,  $\tilde{\phi}(x)$  is the same as in the CGHS model, so that (17) holds. Combining with (34) we find  $\phi^2 - \frac{1}{\beta\phi^2} = 2A$ , where we have again taken  $C = -1$ . Equivalently,  $\phi^4 - 2A\phi^2 - \frac{1}{\beta} = 0$ . This is easily solved — that is what makes the choice  $\alpha = 4$  so nice. We find

$$\phi^2 = A + \sqrt{\frac{1}{\beta} + A^2}, \quad (35)$$

where we chose the solution of the quadratic equation that allowed  $\phi$  to go over to  $\phi_{\text{cghs}}$  in the  $\beta \rightarrow \infty$  limit.

Calculating the scalar curvature is now just a matter of plugging this into (28). A simple but tedious calculation now gives

$$R = \sqrt{2} \lambda^2 \left( \frac{1}{\beta} + A^2 \right)^{-\frac{7}{4}} \left( A + \sqrt{\frac{1}{\beta} + A^2} \right)^{\frac{1}{2}} \cdot \left\{ \frac{16}{\beta \lambda^2} + \frac{3}{\beta} A - \frac{8}{\lambda^2} A^2 + \left( \frac{1}{\beta} - \frac{8}{\lambda^2} A \right) \sqrt{\frac{1}{\beta} + A^2} \right\}, \quad (36)$$

where  $A(x)$  was given in (24). For  $\beta \rightarrow \infty$  we indeed find that

$$R \rightarrow -\frac{32}{A}, \quad (37)$$

which is the CGHS result. From (36) we see that the curvature of the deformed CGHS model is indeed not singular.

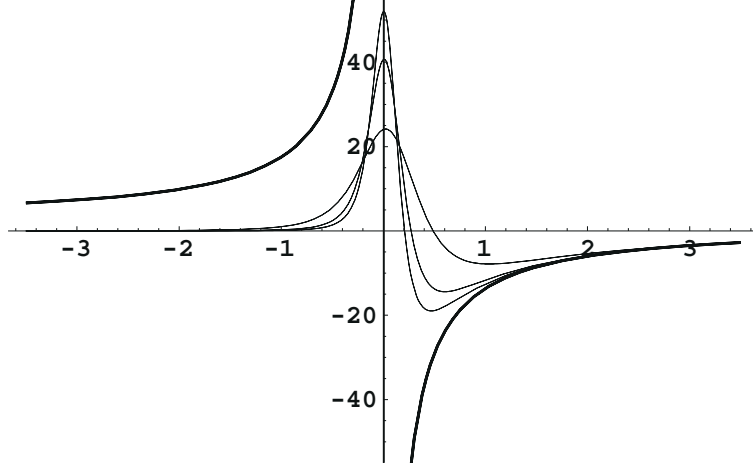


Figure 1:  $R(x)$  for the CGHS model (thick line) and its deformation for  $\beta = 1, 3$  and  $5$ . As  $\beta$  increases the deformations look more and more like the CGHS result (for  $x > 0$ ). The graphs have been plotted for  $\lambda^2 = 1$

As may be seen in Figure 1, the deformed model has maximal curvature at  $x = 0$ . Its value is

$$R_{\max} = \sqrt{2} \left( 16\beta^{\frac{1}{2}} + \lambda^2 \right). \quad (38)$$

At right infinity the deformed model tends to the CGHS result. On the other hand, at left infinity both the CGHS model and its deformation tend to a de Sitter space  $R = \Lambda$ . However, for CGHS we have  $\Lambda = 4\lambda^2$ , while for the deformed model the constant is a complicated function of  $\beta$  and  $\lambda$ .



Rather than writing it out let us only give the result for large  $\beta$  when we have  $\Lambda = 2^{-10}\lambda^8\beta^{-\frac{1}{2}}$ . We have just determined that

$$\lim_{x \rightarrow -\infty} \lim_{\beta \rightarrow \infty} R \neq \lim_{\beta \rightarrow \infty} \lim_{x \rightarrow -\infty} R . \quad (39)$$

Put another way: imposing that our model joins to CGHS at right infinity doesn't automatically guarantee a similar joining at left infinity.

We are now in the position of trying to interpret the meaning of our deformed CGHS model. Obviously, one possibility is to think of (33) as the classical action of a model with scale  $\frac{1}{\beta}$ . However, it seems more natural to interpret our model as an effective action.  $\frac{1}{\beta}$  then naturally comes about from quantization, while  $\beta \rightarrow \infty$  corresponds to the semi classical limit. Our model should thus be the effective action corresponding to the quantization of the CGHS model. Quantization gives  $S \sim \hbar$ , and essentially dimensional analysis (in units  $G = c = 1$ ) gives  $\phi^2 \sim \hbar$ , as well as  $\frac{1}{\beta} \sim \hbar^2$ . Therefore, if we are to interpret our model as an effective action then  $\beta = \kappa \hbar^{-2}$ , where  $\kappa$  is a constant of the order of unity. We see then that the maximal curvature (38) is proportional to  $\frac{1}{\hbar}$ , i.e. represents a non-perturbative effect. Expanding our model in  $\hbar$  we find

$$S_{\text{eff}} = S_{\text{cghs}} - \frac{1}{8\kappa} \hbar^2 \int d^2x \sqrt{-g} (R - 2\lambda^2) \phi^{-2} + o(\hbar^4) . \quad (40)$$

The leading correction to CGHS is of the form of the Jackiw-Teitelboim action for 2d gravity. It would be very interesting to get this result by quantizing some fundamental 2d theory. To do this we would need to start from the CGHS model coupled to some matter fields. We would then have to integrate out the matter. The last step would be to calculate the effective action. It is probably impossible to do this exactly, however, we could hope to do this perturbatively and compare with (40).

## 4 Conclusion

We have constructed a class of exactly solvable 2d gravity models that represent deformations of the CGHS dilaton gravity model. In the semi-classical limit these effective theories go over into the CGHS model. The deformed CGHS models lead to non-singular black hole solutions — i.e. horizons without singularities. We leave a detailed discussion of the global character of the deformed CGHS solutions for a later publication.

It will be interesting to apply this method to non-singular 2d cosmology models. A further avenue of research is to consider deformed models for dilaton gravity in the presence of matter.

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